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ON STRETCHING, TWISTING, PURE BENDING AND FLEXURE OF PRETWISTED ELASTIC PLATES†

E. REISSNER

Department of the Aerospace and Mechanical Engineering Sciences, University of California at San Diego, La Jolla, California

and

F. Y. M. WAN

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts

Abstract—Previously derived results for the mentioned canonical load cases for pretwisted rectangular plates (or shallow helicoidal shells) of rectangular cross section are generalized so as to include arbitrarily prescribed cross wise thickness variations. The present method of derivation is simpler than the earlier one in terms of Marguerre's equations for transverse deflection and Airy stress function, and consists in applying a semi-inverse solution procedure to the system of equilibrium and compatibility equations of shallow shell theory. A particular case of the results of this paper concerns the problem of pure bending of a pretwisted plate with elliptical thickness variation. The results for this case coincide with recent results obtained by Goodier and Griffin through use of three-dimensional elasticity theory.

INTRODUCTION

SOME years ago, Maunder and one of the present authors [4] considered the problem of pure bending of pretwisted bars of narrow rectangular cross section as a problem of the theory of shallow helicoidal shells of uniform thickness. The principal results of this study concerned the effect of pretwist on the magnitudes of the bending stress and of the center line curvature, in comparison with the magnitude of these quantities according to "elementary beam theory". It was found that for beam sections thin enough to justify the use of thin-shell theory the effect of pretwist can be significant—20 per cent or more.

Later on the second named author undertook an analogous investigation of the problem of St. Venant flexure [6], for which the effect of pretwist comes out to be of still greater significance than for the problem of pure bending.

A recent study by Goodier and Griffin is concerned with the problem of pure bending of pretwisted beams by means of an expansion procedure in powers of a (small) pretwist parameter for the equations of three-dimensional elasticity theory [2]. The principal conclusion of this work is that for the effect of pretwist to be numerically significant, the cross section of the beam must be "thin". Goodier and Griffin present results for a beam with thin *elliptical* cross section and state that the results for this case are "comparable" with the results of Maunder and Reissner for the rectangular cross section case.

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A study of the analysis by Goodier and Griffin indicates, in conformity with the intuitive feeling which led to the work in [4], that the effect of pretwist is significant for just that range of parameter values for which application of the theory of thin shallow shells (or "*pretwisted* plates") is appropriate. As an analysis of the problem by means of the theory of shallow helicoidal shells is a great deal simpler than the corresponding analysis by means of the equations of three-dimensional elasticity theory we extend in what follows the previously undertaken shallow-shell-theoretical approach for uniform pretwisted plates to the problems of stretching, twisting, bending and flexure of plates with thickness and material properties variable in crosswise direction.

Insofar as the derivation of suitable solutions of the equations for shallow helicoidal shells is concerned, we find it convenient here not to start with the usual two simultaneous shallow-shell differential equations for the transverse deflection w and an Airy stress function F as in our earlier work on shells with uniform properties [4, 6], but rather to subject directly to a semi-inverse procedure of solution the system of equilibrium, compatibility and constitutivity differential equations which have recently been stated in [5].

Among the explicit results which are obtained in the following we mention, in particular, formulas for bending stress and centerline curvature in pretwisted plates with thickness varying elliptically in crosswise direction, which are *identical* with the formulas given by Goodier and Griffin on the basis of their asymptotic expansion procedure for the three-dimensional problem.

FORMULATION OF THE PROBLEM

We consider a shallow shell with middle surface equation

$$z = \beta xy \tag{1}$$

for $|x| \leq a$ and $|y| \leq b$, and take the differential equations of shallow shell theory in the form stated in [5], specialized by the assumption of absent surface forces, moment stress couples and transverse shear deformations. We then have as equations of equilibrium

$$N_{xx,x}+N_{yx,y} = 0, \qquad N_{xy,x}+N_{yy,y} = 0, \tag{2}$$

$$Q_{x,x}+Q_{y,y}+\beta(N_{xy}+N_{yx}) = 0, \tag{3}$$

$$M_{xx,x}+M_{yx,y} = Q_x, \qquad M_{xy,x}+M_{yy,y} = Q_y, \tag{4}$$

where $N_{xy} = N_{yx}$, and as equations of compatibility

$$\varkappa_{yy,x}-\varkappa_{xy,y} = 0, \qquad \varkappa_{yx,x}-\varkappa_{xx,y} = 0, \tag{5}$$

$$\lambda_{y,x}-\lambda_{x,y}+\beta(\varkappa_{xy}+\varkappa_{yx}) = 0, \tag{6}$$

$$\varepsilon_{yy,x}-\varepsilon_{xy,y} = \lambda_y, \qquad \varepsilon_{yx,x}-\varepsilon_{xx,y} = \lambda_x, \tag{7}$$

where $\varkappa_{xy} = \varkappa_{yx}$.

We stipulate that the edges $y = \pm b$ are free of traction, that is, we prescribe the boundary conditions

$$y = \pm b: \qquad N_{yy} = N_{yx} = Q_y + M_{yx,x} = M_{yy} = 0. \tag{8}$$

In prescribing boundary conditions for the edges $x = \pm a$, we assume that they are acted upon by given forces and moments and we consider separately:

(i) The case of stretching, twisting and pure bending for which the boundary conditions are

$$x = \pm a \begin{cases} \int_{-b}^{b} N_{xx}\,\mathrm{d}y = N, \qquad \int_{-b}^{b} N_{xy}\,\mathrm{d}y = 0, & (9) \\ \int_{-b}^{b} (Q_x + M_{xy,y} + z_{,x}N_{xx} + z_{,y}N_{xy})\,\mathrm{d}y - [2M_{xy}]_{-b}^{b} = 0, & (10) \\ \int_{-b}^{b} yN_{xx}\,\mathrm{d}y = M_s, \qquad \int_{-b}^{b} (M_{xx} + zN_{xx})\,\mathrm{d}y = M_p, & (11) \\ \int_{-b}^{b} (Q_x + M_{xy,y} + z_{,x}N_{xx} + z_{,y}N_{xy})y\,\mathrm{d}y - [2yM_{xy}]_{-b}^{b} = M_t. & (12) \end{cases}$$

(ii) The case of flexure due to equal and opposite forces in the directions of y and z, for which the boundary conditions are

$$x = \pm a \begin{cases} \int_{-b}^{b} N_{xx}\,\mathrm{d}y = 0, \qquad \int_{-b}^{b} N_{xy}\,\mathrm{d}y = Q_s, & (13) \\ \int_{-b}^{b} (Q_x + M_{xy,y} + z_{,x}N_{xx} + z_{,y}N_{xy})\,\mathrm{d}y - [2M_{xy}]_{-b}^{b} = Q_p, & (14) \\ \int_{-b}^{b} yN_{xx}\,\mathrm{d}y = \pm Q_s a, \qquad \int_{-b}^{b} (M_{xx} + zN_{xx})\,\mathrm{d}y = \pm Q_p a, & (15) \\ \int_{-b}^{b} [y(Q_x + M_{xy,y} + z_{,x}N_{xx} + z_{,y}N_{xy}) - zN_{xy}]\,\mathrm{d}y - [2yM_{xy}]_{-b}^{b} = 0 & (16) \end{cases}$$

The subscripts s and p indicate sheet and plate action respectively.

The foregoing system of differential equations and boundary conditions is supplemented by constitutive equations which are here taken in the form

$$C\varepsilon_{xx} = N_{xx} - \nu_N N_{yy}, \qquad C\varepsilon_{yy} = N_{yy} - \nu_N N_{xx}, \qquad C\varepsilon_{xy} = C\varepsilon_{yx} = (1+\nu_N)N_{xy}, \tag{17}$$

$$M_{xx} = D(\varkappa_{xx} + \nu_M \varkappa_{yy}), \qquad M_{yy} = D(\varkappa_{yy} + \nu_M \varkappa_{xx}), \qquad M_{xy} = M_{yx} = (1-\nu_M)D\varkappa_{xy}. \tag{18}$$

In these *the coefficients* C, D, ν_N *and* ν_M *are given functions of* y.

STRETCHING, TWISTING AND PURE BENDING

Guided by the form of the boundary conditions (9)–(12), we attempt a solution of the problem through a semi-inverse procedure, assuming at the outset a state of stress with the properties that

1. N_{xy}, N_{yy}, Q_y, M_{yy} vanish throughout..
2. N_{xx}, Q_x, $M_{xy} = M_{yx}$ are independent of x,
3. $M_{xx} = M_0(y) + xM_1(y)$.

With the above assumptions, all but one of the equilibrium equations (2)–(4) are satisfied automatically, leaving only the relation

$$Q_x = M^{\cdot}_{yx}(y) + M_1(y), \tag{19}$$

with differentiation with respect to y from now on indicated by dots. Furthermore, the boundary conditions (8) for $y = \pm b$ are satisfied automatically.

Introduction of assumptions (1)–(3) into the constitutive equations (17) and (18) reduces these to the form

$$C\varepsilon_{xx} = N_{xx}(y), \qquad \varepsilon_{yy} = -\nu_N \varepsilon_{xx}, \qquad \varepsilon_{xy} = \varepsilon_{yx} = 0, \tag{20}$$

$$\begin{gathered}(1-\nu_M^2)D\varkappa_{xx} = M_0(y) + xM_1(y), \qquad \varkappa_{yy} = -\nu_M \varkappa_{xx}, \\ (1-\nu_M)D\varkappa_{xy} = (1-\nu_M)D\varkappa_{yx} = M_{xy}(y).\end{gathered} \tag{21}$$

Introduction of (20) and (21) into the compatibility equations (5) and (7) leaves the relations

$$\frac{\nu_M M_1(y)}{D(1-\nu_M^2)} + \left[\frac{M_{xy}(y)}{D(1-\nu_M)}\right]^{\cdot} = 0, \tag{22}$$

$$\left[\frac{M_0(y)}{D(1-\nu_M^2)}\right]^{\cdot} = 0, \qquad \left[\frac{M_1(y)}{D(1-\nu_M^2)}\right]^{\cdot} = 0, \tag{23}$$

$$\lambda_x = -\left[\frac{N_{xx}(y)}{C}\right]^{\cdot}, \qquad \lambda_y = 0 \tag{24}$$

and, on the basis of (6),

$$\left[\frac{N_{xx}(y)}{C}\right]^{\cdot\cdot} + \frac{2\beta M_{xy}(y)}{D(1-\nu_M)} = 0. \tag{25}$$

We now have from (23)

$$M_0(y) = c_1(1-\nu_M^2)D, \qquad M_1(y) = c_2(1-\nu_M^2)D \tag{26}$$

where c_1 and c_2 are constants of integration, and from (22)

$$M_{xy} = -\left(c_2\int_0^y \nu_M \, dy + c_3\right)(1-\nu_M)D. \tag{27}$$

Introduction of (27) into (25) gives

$$N_{xx} = \left\{2\beta\left[c_2\int_0^y\int_0^y\int_0^y \nu_M \, dy \, dy \, dy + \tfrac{1}{2}c_3 y^2\right] + c_4 y + c_5\right\}C. \tag{28}$$

Combination of (26)–(28) with (20) and (21) gives expressions for all components of strains in terms of the given function C, D, ν_M and ν_N, and of the five constants of integration c_i.

Corresponding expressions for all non-vanishing stress resultants and couples are given by (27) and (28), together with

$$M_{xx} = (c_1 + c_2 x)(1-\nu_M^2)D, \tag{29}$$

and

$$Q_x = c_2\left[(1-\nu_M^2)D - \left\{(1-\nu_M)D\int_0^y \nu_M \,\mathrm{d}y\right\}^{\cdot}\right] - c_3[(1-\nu_M)D]^{\cdot} \tag{30}$$

It remains to satisfy the boundary condition (9)–(12) for $x = \pm a$. Of these, one is satisfied automatically and the remaining five become five simultaneous equations for the five constants of integration c_i. In stating these five equations, we shall for simplicity's sake assume $\nu_M = \text{const}$. We then have, from (9)

$$\frac{1}{3}c_2\beta\nu_M\int_{-b}^{b} y^3C\,\mathrm{d}y + c_3\beta\int_{-b}^{b} y^2C\,\mathrm{d}y + c_4\int_{-b}^{b} yC\,\mathrm{d}y + c_5\int_{-b}^{b} C\,\mathrm{d}y = N. \tag{31}$$

Equation (10) can be shown to be implied by the second relation in both (9) and (11) in conjunction with (19). The first equation in (11) becomes

$$\frac{1}{3}c_2\beta\nu_M\int_{-b}^{b} y^4C\,\mathrm{d}y + c_3\beta\int_{-b}^{b} y^3C\,\mathrm{d}y + c_4\int_{-b}^{b} y^2C\,\mathrm{d}y + c_5\int_{-b}^{b} yC\,\mathrm{d}y = M_s. \tag{32}$$

From the second equation in (11) follow *two* relations,

$$c_1(1-\nu_M^2)\int_{-b}^{b} D\,\mathrm{d}y = M_p, \qquad c_2(1-\nu_M^2)\int_{-b}^{b} D\,\mathrm{d}y = -\beta M_s. \tag{33a, b}$$

Finally, (12) gives

$$\begin{aligned} c_2\left[\frac{1}{3}\beta^2\nu_M\int_{-b}^{b} y^5C\,\mathrm{d}y + (1-\nu_M)(1+3\nu_M)\int_{-b}^{b} yD\,\mathrm{d}y\right] \\ + c_3\left[\beta^2\int_{-b}^{b} y^4C\,\mathrm{d}y + 2(1-\nu_M)\int_{-b}^{b} D\,\mathrm{d}y\right] \\ + c_4\beta\int_{-b}^{b} y^3C\,\mathrm{d}y + c_5\beta\int_{-b}^{b} y^2C\,\mathrm{d}y = M_t. \end{aligned} \tag{34}$$

It is apparent that with c_1 and c_2 directly given by (33), equations (31), (32) and (34) become three simultaneous equations for the determination of c_3, c_4 and c_5, with coefficients depending on the section property integrals $\int_{-b}^{b} y^mC\,\mathrm{d}y$ for $m = 0, 1, 2, 3, 4, 5$ and $\int_{-b}^{b} y^mD\,\mathrm{d}y$, for $m = 0, 1$.

In terms of these constants, we have M_{xx} as in (29), while from (27) and (28)

$$M_{xy} = -(c_3 + c_2\nu_M y)(1-\nu_M)D, \tag{35}$$

$$N_{xx} = (c_5 + c_4y + c_3\beta y^2 + \tfrac{1}{3}c_2\beta\nu_M y^3)C \tag{36}$$

Furthermore, in accordance with (20), (21), (26), (35) and (36),

$$\varkappa_{xx} = c_1 + c_2x, \qquad \varkappa_{xy} = -c_3 - c_2\nu_M y, \tag{37}$$

$$\varepsilon_{xx} = c_5 + c_4y + c_3\beta y^2 + \tfrac{1}{3}c_2\beta\nu_M y^3 \tag{38}$$

while ε_{yy}, ε_{xy}, ε_{yx} and $\varkappa_{yy}$ are as in (20) and (21).

Having (37), (38), (20) and (21), we may readily calculate displacements in accordance with the formulas

$$u_{x,x} = \varepsilon_{xx}, \qquad u_{x,y}+u_{y,x}-2\beta w = \varepsilon_{xy}+\varepsilon_{yx}, \qquad u_{y,y} = \varepsilon_{yy},$$
$$w_{,xx} = -\varkappa_{xx}, \qquad w_{,xy} = -\varkappa_{xy}, \qquad w_{,yy} = -\varkappa_{yy}. \tag{39}$$

THE SYMMETRIC CROSS SECTION CASE

The foregoing results are simplified considerably for the case that the cross section property functions C and D are even in y so that

$$\int_{-b}^{b} (y, y^3, y^5)C \, \mathrm{d}y = 0, \qquad \int_{-b}^{b} yD \, \mathrm{d}y = 0. \tag{40}$$

We then obtain from (32), with the help of (33b)

$$c_4 = \frac{M_s}{\int_{-b}^{b} y^2 C \, \mathrm{d}y}\left[1+\frac{1}{3}\frac{\nu_M \beta^2}{1-\nu_M^2}\frac{\int_{-b}^{b} y^4 C \, \mathrm{d}y}{\int_{-b}^{b} D \, \mathrm{d}y}\right] \tag{41}$$

and from (31) and (34)

$$c_3 = \frac{\dfrac{M_t}{2(1-\nu_M)\int_{-b}^{b} D \, \mathrm{d}y} - \dfrac{\int_{-b}^{b} y^2 C \, \mathrm{d}y}{2(1-\nu_M)\int_{-b}^{b} D \, \mathrm{d}y}\dfrac{\beta N}{\int_{-b}^{b} C \, \mathrm{d}y}}{1+\dfrac{\beta^2 \int_{-b}^{b} y^4 C \, \mathrm{d}y}{2(1-\nu_M)\int_{-b}^{b} D \, \mathrm{d}y}\left[1-\dfrac{(\int_{-b}^{b} y^2 C \, \mathrm{d}y)^2}{(\int_{-b}^{b} y^4 C \, \mathrm{d}y)(\int_{-b}^{b} C \, \mathrm{d}y)}\right]}. \tag{42}$$

$$c_5 = \frac{\left(1+\dfrac{\beta^2 \int_{-b}^{b} y^4 C \, \mathrm{d}y}{2(1-\nu_M)\int_{-b}^{b} D \, \mathrm{d}y}\right)\dfrac{N}{\int_{-b}^{b} C \, \mathrm{d}y} - \dfrac{\int_{-b}^{b} y^2 C \, \mathrm{d}y}{\int_{-b}^{b} C \, \mathrm{d}y}\dfrac{\beta M_t}{2(1-\nu_M)\int_{-b}^{b} D \, \mathrm{d}y}}{1+\dfrac{\beta^2 \int_{-b}^{b} y^4 C \, \mathrm{d}y}{2(1-\nu_M)\int_{-b}^{b} D \, \mathrm{d}y}\left[1-\dfrac{(\int_{-b}^{b} y^2 C \, \mathrm{d}y)^2}{(\int_{-b}^{b} y^4 C \, \mathrm{d}y)(\int_{-b}^{b} C \, \mathrm{d}y)}\right]}. \tag{43}$$

The following quantities are of particular interest:

Axial extension and twist due to applied axial torque M_t and axial force N

From equations (37) and (38) follows

$$\varepsilon_{xx}(x, y) = c_5 + c_3 \beta y^2, \qquad \varkappa_{xy}(x, y) = -c_3, \tag{44}$$

where c_5 and c_3 are given by (43) and (42). As far as the authors know, the results of equation (44) have not previously been given, except for the case of a rectangular cross section [3], and for the case $N = 0$ [1]. In accordance with (20) and (22), we have further that $\sigma_{xx} = (C/h)\varepsilon_{xx}$, uniform across the thickness, and $\sigma_{xy} = \pm[6(1-\nu_M)D/h^2]\varkappa_{xy}$ for the two face surfaces of the shell.

Pure plate bending due to moment M_p

Setting $M_s = 0$, $M_t = 0$ and $N = 0$, we have

$$\varkappa_{xx}(x, y) = c_1 \tag{45}$$

there being no effect of pretwist to the degree of approximation implied by shallow shell theory, as previously found for the case of a rectangular cross section [4].

Pure sheet bending due to moment M_s

Setting $M_p = 0$, $M_t = 0$ and $N = 0$, we have

$$\varkappa_{xx}(x, y) = c_2 x, \qquad \varkappa_{xy}(x, y) = -c_2 \nu_M y. \tag{46}$$

$$\varepsilon_{xx}(x, y) = \tfrac{1}{3} c_2 \beta \nu_M y^3 + c_4 y. \tag{47}$$

For this problem we are interested, in particular, in the values of the stress σ_{xx}^D for $y = \pm b$, given by

$$\sigma_{xx}^D(x, \pm b) = \left(\frac{N_{xx}}{h}\right)_{y=\pm b} = \pm\frac{C}{h}(\tfrac{1}{3}c_2\beta\nu_M b^3 + c_4 b). \tag{48}$$

and in the values of the curvature component k_y of the deflected centerline of the shell, given by $(u_y - z_{,y}w)_{,xx}$ for $y = 0$. Use of equations (39) leads to the relation

$$k_y = -\varepsilon_{xx,y}(x, 0) + \beta x \varkappa_{xx}(x, 0) \tag{49}$$

in accordance with what has been stated in [4]. From (46) and (47) follows as expression for k_y in terms of the coefficients c_2 and c_4

$$k_y = -c_4 + c_2 \beta x^2. \tag{50}$$

The term quadratic in x is the same as that which follows from elementary beam theory if account is taken of the rotation of the principal axes of the cross section. The term with c_4 incorporates the Poisson's ratio effect of pretwist, which is not given by beam theory and which has earlier been obtained for the special case of a rectangular cross section [4].

Equations (48) and (50) contain the results which are of particular concern here. We may write them more explicitly as

$$\frac{\sigma_{xx}(x, \pm b)}{\sigma_0} = 1 - \rho_\sigma, \qquad \rho_\sigma = \frac{\nu_M \beta^2}{1-\nu_M^2} \frac{\int_{-b}^{b} (b^2 - y^2) y^2 C \, dy}{3 \int_{-b}^{b} D \, dy}, \tag{51}$$

$$\frac{k_y}{k_0} = 1 + \rho_k + \frac{\beta^2 x^2 \int_{-b}^{b} y^2 C \, dy}{(1-\nu_M^2)\int_{-b}^{b} D \, dy}, \qquad \rho_k = \frac{\nu_M \beta^2}{1-\nu_M^2} \frac{\int_{-b}^{b} y^4 C \, dy}{3 \int_{-b}^{b} D \, dy}, \tag{52}$$

where

$$\sigma_0 = \frac{\pm M_s}{hb^2} \frac{b^3 C}{\int_{-b}^{b} y^2 C \, dy}, \qquad k_0 = -\frac{M_s}{\int_{-b}^{b} y^2 C \, dy}. \tag{53}$$

To see the effect of the cross section shape in the problem of pure sheet bending, we consider, with

$$D = \frac{E_M h^3}{12(1-\nu_M^2)}, \qquad C = E_N h \tag{54}$$

the cases of rectangular, elliptical, lenticular and diamond shaped cross sections. The following results are obtained:

$$
\begin{aligned}
&\frac{h}{h_0} = 1; && \rho_\sigma = \frac{8}{15}\frac{\nu_M E_N \beta^2 b^4}{E_M h_0^2}, && \rho_k = \frac{4}{5}\frac{\nu_M E_N \beta^2 b^4}{E_M h_0^2},\\
&\frac{h}{h_0} = \sqrt{\left(1-\frac{y^2}{b^2}\right)}; && \rho_\sigma = \frac{2}{3}\frac{\nu_M E_N \beta^2 b^4}{E_M h_0^2}, && \rho_k = \frac{2}{3}\frac{\nu_M E_N \beta^2 b^4}{E_M h_0^2},\\
&\frac{h}{h_0} = 1-\frac{y^2}{b^2}; && \rho_\sigma = \frac{7}{6}\frac{\nu_M E_N \beta^2 b^4}{E_M h_0^2}, && \rho_k = \frac{1}{2}\frac{\nu_M E_N \beta^2 b^4}{E_M h_0^2},\\
&\frac{h}{h_0} = 1-\frac{|y|}{b}; && \rho_\sigma = \frac{4}{5}\frac{\nu_M E_N \beta^2 b^4}{E_M h_0^2}, && \rho_k = \frac{8}{15}\frac{\nu_M E_N \beta^2 b^4}{E_M h_0^2}.
\end{aligned}
\tag{55}
$$

When $E_M = E_N$ the results for the case $h/h_0 = 1$ coincide with those given in [4] and the results for the case $h/h_0 = (1-y^2/b^2)^{\frac{1}{2}}$ coincide with those derived by Goodier and Griffin [2].

ST. VENANT FLEXURE DUE TO TRANSVERSE END FORCES

Since there is no coupling between the problems of St. Venant flexure due to plate forces $\pm Q_p$ and sheet forces $\pm Q_s$, we consider the two problems separately.

The form of the boundary conditions (13)–(16) with $Q_s = 0$ suggests that we assume at the outset the same state of stress as that of the section dealing with stretching, twisting and pure bending, with $M_0(y) = 0$. With $\nu_M =$ const., the previous reduction gives the following expressions for the nonvanishing stress resultants and couples:

$$
\begin{aligned}
N_{xx} &= C\{\tfrac{1}{3}c_2\nu_M\beta y^3 + c_3\beta y^2 + c_4 y + c_5\},\\
Q_x &= c_2\{(1-\nu_M^2)D - [(1-\nu_M)\nu_M D y]^{\boldsymbol{\cdot}}\} - c_3\{(1-\nu_M)D\}^{\boldsymbol{\cdot}},\\
M_{xy} &= -(c_2\nu_M y + c_3)(1-\nu_M)D, \qquad M_{xx} = c_2 x(1-\nu_M^2)D,
\end{aligned}
\tag{56}
$$

with the strain and curvature change measures being given by equations (20) and (21).

The boundary conditions at $y = \pm b$, (8), are again automatically satisfied. Of the six boundary conditions at $x = \pm a$ with $Q_s = 0$, the second condition in (13) is satisfied automatically. The first conditions in (13) and (15) become

$$
\frac{1}{3}c_2\nu_M\beta\int_{-b}^{b} y^3 C\,\mathrm{d}y + c_3\beta\int_{-b}^{b} y^2 C\,\mathrm{d}y + c_4\int_{-b}^{b} yC\,\mathrm{d}y + c_5\int_{-b}^{b} C\,\mathrm{d}y = 0,
\tag{57}
$$

$$
\frac{1}{3}c_2\nu_M\beta\int_{-b}^{b} y^4 C\,\mathrm{d}y + c_3\beta\int_{-b}^{b} y^3 C\,\mathrm{d}y + c_4\int_{-b}^{b} y^2 C\,\mathrm{d}y + c_5\int_{-b}^{b} yC\,\mathrm{d}y = 0.
\tag{58}
$$

With the first condition in (15), equation (14) determines c_2 in terms of Q_p,

$$
c_2(1-\nu_M^2)\int_{-b}^{b} D\,\mathrm{d}y = Q_p,
\tag{59}
$$

and the second condition in (15) is satisfied without further restrictions imposed on the c_i. Finally, condition (16) becomes

$$c_2\left[(1+3\nu_M)(1-\nu_M)\int_{-b}^{b} yD\,\mathrm{d}y+\frac{1}{3}\nu_M\beta^2\int_{-b}^{b} y^5C\,\mathrm{d}y\right]$$
$$+c_3\left[2(1-\nu_M)\int_{-b}^{b} D\,\mathrm{d}y+\beta^2\int_{-b}^{b} y^4C\,\mathrm{d}y\right]$$
$$+\beta c_4\int_{-b}^{b} y^3C\,\mathrm{d}y+\beta c_5\int_{-b}^{b} y^2C\,\mathrm{d}y = 0. \tag{60}$$

With c_2 given directly by (59), equations (57), (58) and (60) become three simultaneous inhomogeneous equations for the determination of c_3, c_4 and c_5. For shells with symmetric cross sections,

$$c_3 = c_5 = 0, \qquad c_4 = -\frac{1}{3}\frac{\nu_M\beta Q_p}{(1-\nu_M^2)\int_{-b}^{b} D\,\mathrm{d}y}\frac{\int_{-b}^{b} y^4C\,\mathrm{d}y}{\int_{-b}^{b} y^2C\,\mathrm{d}y}, \tag{61}$$

and therewith

$$N_{xx} = \frac{1}{3}\frac{\nu_M\beta CQ_p}{(1-\nu_M^2)\int_{-b}^{b} D\,\mathrm{d}y}\left\{y^3-y\frac{\int_{-b}^{b} y^4C\,\mathrm{d}y}{\int_{-b}^{b} y^2C\,\mathrm{d}y}\right\},$$
$$Q_x = \frac{Q_p}{\int_{-b}^{b} D\,\mathrm{d}y}\left\{D-\left[\frac{\nu_M D_y}{1+\nu_M}\right]^{\cdot}\right\},$$
$$M_{xy} = -\frac{\nu_M DQ_p y}{(1+\nu_M)\int_{-b}^{b} D\,\mathrm{d}y}, \qquad M_{xx} = \frac{DQ_p x}{\int_{-b}^{b} D\,\mathrm{d}y}. \tag{62}$$

Upon introducing the appropriate specializations for an isotropic homogeneous shell of constant thickness, these expressions for resultants and couples reduce to those obtained in [6]. They also show that the only effect of pretwist is, to the degree of approximation of shallow shell theory and independent of how D and C vary as functions of y, the generation of the supplementary membrane stress resultant N_{xx} as first observed in [6] for the case of a homogeneous isotropic shell of constant thickness.†

With $Q_p = 0$, the boundary conditions (13)–(16) suggest a semi-inverse solution based on the assumptions

1. N_{yy} vanishes throughout,
2. N_{xy}, Q_y and M_{yy} are independent of x,
3. $N_{xx} = xN_1(y)$, $\quad Q_x = xQ_1(y)$, $\quad M_{xy} = xT(y)$,
4. $M_{xx} = M_0(y)+\frac{1}{2}x^2M_2(y)$.

Upon introducing these into the equilibrium equations (2)–(4), we have

$$N_{xx} = -xN_{xy}^{\cdot}, \qquad Q_x = -x(Q_y^{\cdot}+2\beta N_{xy}),$$
$$M_{xy} = x(Q_y-M_{yy}^{\cdot}), \qquad M_{xx} = M_0(y)+\tfrac{1}{2}x^2(M_{yy}^{\cdot\cdot}-2Q_y^{\cdot}-2\beta N_{xy}) \tag{63}$$

with the four unknowns N_{xy}, Q_y, M_{yy} and M_0 to be determined as functions of y by the remaining differential equations and boundary conditions.

† It can also be verified with the help of (39) that the transverse displacement component w is unaffected by the pretwist and that there will be tangential displacement components u_x and u_y generated by the presence of the pretwist.

Substitution of (63) into the stress strain relations (17) and (18) leads to the following expressions for the mid-surface strain and curvature change measures

$$
\begin{aligned}
C\varepsilon_{xx} &= -xN^{\cdot}_{xy}, \qquad \varepsilon_{yy} = -\nu_N\varepsilon_{xx}, \qquad C\varepsilon_{xy} = C\varepsilon_{yx} = (1+\nu_N)N_{xy}, \\
(1-\nu_M^2)D\varkappa_{xx} &= (M_0-\nu_M M_{yy})+\tfrac{1}{2}x^2(M^{\cdot\cdot}_{yy}-2Q^{\cdot}_y-2\beta N_{xy}), \\
(1-\nu_M^2)D\varkappa_{yy} &= (M_{yy}-\nu_M M_0)-\tfrac{1}{2}\nu_M x^2(M^{\cdot\cdot}_{yy}-2Q^{\cdot}_y-2\beta N_{xy}), \\
(1-\nu_M)D\varkappa_{xy} &= (1-\nu_M)D\varkappa_{yx} = x(Q_y-M^{\cdot}_{yy}).
\end{aligned}
\tag{64}
$$

Introduction of (64) into the compatibility equations (5)–(7) leaves, with a constant of integration c_1, the following four ordinary differential equations for the determination of the four unknown functions of y:

$$
\begin{aligned}
&\left[\frac{Q_y-M^{\cdot}_{yy}}{D(1-\nu_M)}\right]^{\cdot}+\nu_M c_1 = 0, \qquad \left(\frac{N^{\cdot}_{xy}}{C}\right)^{\cdot\cdot}+\frac{2\beta(M^{\cdot}_{yy}-Q_y)}{D(1-\nu_M)} = 0, \\
&M^{\cdot\cdot}_{yy}-2Q^{\cdot}_y-2\beta N_{xy} = c_1 D(1-\nu_M^2), \qquad \left[\frac{M_0-\nu_M M_{yy}}{D(1-\nu_M^2)}\right]^{\cdot}+\frac{M^{\cdot}_{yy}-Q_y}{D(1-\nu_M)} = 0.
\end{aligned}
\tag{65}
$$

Equations (65) have the solution

$$
N_{xy} = -\int_{-b}^{y}(\tfrac{1}{3}c_1\nu_M\beta y^3+c_2\beta y^2-c_3y-c_4)C\,\mathrm{d}y, \tag{66}
$$

$$
Q_y = D(1-\nu_M)\left[c_1\left(\nu_M y-\frac{1+\nu_M}{D}\int_{-b}^{y}D\,\mathrm{d}y\right)+c_2\right]-2\beta\int_{-b}^{y}N_{xy}\,\mathrm{d}y, \tag{67}
$$

$$
M_{yy} = c_1\nu_M(1-\nu_M)\int_{-b}^{y}yD\,\mathrm{d}y+c_2(1-\nu_M)\int_{-b}^{y}D\,\mathrm{d}y+\int_{-b}^{y}Q_y\,\mathrm{d}y, \tag{68}
$$

$$
M_0 = \nu_M M_{yy}-D(1-\nu_M^2)(\tfrac{1}{2}c_1\nu_M y^2+c_2y+c_5) \tag{69}
$$

where c_2–c_5 are additional constants of integration and where we have again assumed for simplicity's sake that ν_M is a constant. We have also chosen the lower limit of integration of the various integrals such that $M_{yy}(-b) = N_{xy}(-b) = 0$.

With the above results, we then have the following expressions for stress resultants and couples

$$
\begin{aligned}
N_{yy} &= 0, \qquad N_{xx} = Cx(\tfrac{1}{3}c_1\nu_M\beta y^3+c_2\beta y^2-c_3y-c_4), \\
N_{xy} &= -\int_{-b}^{y}(\tfrac{1}{3}c_1\nu_M\beta y^3+c_2\beta y^2-c_3y-c_4)C\,\mathrm{d}y, \\
Q_x &= (1-\nu_M)x[c_1(D-\nu_M D^{\cdot}y)-c_2D^{\cdot}], \\
Q_y &= (1-\nu_M)\left\{c_1\left[\nu_M Dy-(1+\nu_M)\int_{-b}^{y}D\,\mathrm{d}y\right]+c_2D\right\}-2\beta\int_{-b}^{y}N_{xy}\,\mathrm{d}y, \\
M_{xy} &= -D(1-\nu_M)x(c_1\nu_M y+c_2), \\
M_{yy} &= c_1\nu_M(1-\nu_M)\int_{-b}^{y}yD\,\mathrm{d}y+c_2(1-\nu_M)\int_{-b}^{y}D\,\mathrm{d}y+\int_{-b}^{y}Q_y\,\mathrm{d}y, \\
M_{xx} &= \nu_M M_{yy}-D(1-\nu_M^2)[\tfrac{1}{2}c_1(\nu_M y^2-x^2)+c_2y+c_5].
\end{aligned}
\tag{70}
$$

All the boundary conditions (8) for $y = \pm b$ and (13)–(16) for $x = \pm a$ will be satisfied if the constants of integration, c_1–c_5 are determined by the following set of equations

$$c_1 = -\frac{2\beta Q_s}{(1-\nu_M^2)\int_{-b}^{b} D\,dy}, \tag{71}$$

$$-\beta c_2\int_{-b}^{b} y^2 C\,dy + c_3\int_{-b}^{b} yC\,dy + c_4\int_{-b}^{b} C\,dy = -\frac{1}{3}\nu_M\beta c_1\int_{-b}^{b} y^3 C\,dy. \tag{72}$$

$$-\beta c_2\int_{-b}^{b} y^3 C\,dy + c_3\int_{-b}^{b} y^2 C\,dy + c_4\int_{-b}^{b} yC\,dy = -Q_s + \frac{1}{3}\nu_M\beta c_1\int_{-b}^{b} y^4 C\,dy, \tag{73}$$

$$\begin{aligned} c_2\left[2(1-\nu_M)\int_{-b}^{b} D\,dy + \beta^2\int_{-b}^{b} y^4 C\,dy\right] - \beta c_3\int_{-b}^{b} y^2 C\,dy - \beta c_4\int_{-b}^{b} y^2 C\,dy \\ = -c_1\left[(1+3\nu_M)(1-\nu_M)\int_{-b}^{b} yD\,dy + \tfrac{1}{3}\nu_M\beta^2\int_{-b}^{b} y^5 C\,dy\right], \end{aligned} \tag{74}$$

$$\begin{aligned} c_5(1-\nu_M^2)\int_{-b}^{b} D\,dy = {} & -c_1\left[\nu_M(1-\nu_M)(1+3\nu_M)\int_{-b}^{b} y^2 D\,dy + \tfrac{1}{9}\nu_M^2\beta^2\int_{-b}^{b} y^6 C\,dy\right] \\ & -c_2\left[(1-\nu_M)(1+3\nu_M)\int_{-b}^{b} yD\,dy + \tfrac{1}{3}\nu_M\beta^2\int_{-b}^{b} y^5 C\,dy\right] \\ & +\tfrac{1}{3}c_3\nu_M\beta\int_{-b}^{b} y^4 C\,dy + \tfrac{1}{3}c_4\nu_M\beta\int_{-b}^{b} y^3 C\,dy. \end{aligned} \tag{75}$$

To indicate the nature of the analysis involved in verifying that the boundary conditions are in fact satisfied, we note, for example, the transformation

$$\begin{aligned} \int_{-b}^{b} N_{xy}\,dy &= -\int_{-b}^{b}\int_{-b}^{y} (\tfrac{1}{3}c_1\nu_M\beta y^3 + c_2\beta y^2 - c_3 y - c_4)C\,dy\,dy \\ &= -\left[y\int_{-b}^{y} (\tfrac{1}{3}c_1\nu_M\beta y^3 + c_2\beta y^2 - c_3 y - c_4)C\,dy\right]_{-b}^{b} \\ &\quad + \int_{-b}^{b} (\tfrac{1}{3}c_1\nu_M\beta y^3 + c_2\beta y^2 - c_3 y - c_4)yC\,dy \\ &= -b\int_{-b}^{b} (\tfrac{1}{3}c_1\nu_M\beta y^3 + c_2\beta y^2 - c_3 y - c_4)C\,dy \\ &\quad + \int_{-b}^{b} (\tfrac{1}{3}c_1\nu_M\beta y^6 + c_2\beta y^2 - c_3 y - c_4)yC\,dy. \end{aligned} \tag{76}$$

The first integral on the right vanishes because of equation (72). What remains is equal to Q_s by equation (73). As such, the second boundary condition in (13) is satisfied.

For shells with D and C being even functions of y, equations (72) and (74) give $c_2 = c_4 = 0$. Equation (73) becomes

$$c_3\int_{-b}^{b} y^2 C\,dy = -Q_s\left\{1 + \frac{2}{3}\frac{\nu_M\beta^2}{1-\nu_M^2}\frac{\int_{-b}^{b} y^4 C\,dy}{\int_{-b}^{b} D\,dy}\right\}, \tag{77}$$

and equation (75) gives

$$c_5(1-\nu_M^2)\int_{-b}^{b} D\,\mathrm{d}y = 2\nu_M\beta Q_s\left\{\frac{1+3\nu_M}{1+\nu_M}\frac{\int_{-b}^{b} y^2 D\,\mathrm{d}y}{\int_{-b}^{b} D\,\mathrm{d}y}+\frac{\nu_M\beta^2}{9(1-\nu_M^2)}\frac{\int_{-b}^{b} y^6 C\,\mathrm{d}y}{\int_{-b}^{b} D\,\mathrm{d}y}\right.$$
$$\left.-\frac{1}{6}\frac{\int_{-b}^{b} y^4 C\,\mathrm{d}y}{\int_{-b}^{b} y^2 C\,\mathrm{d}y}\left[1+\frac{2\nu_M\beta^2}{3(1-\nu_M^2)}\frac{\int_{-b}^{b} y^4 C\,\mathrm{d}y}{\int_{-b}^{b} D\,\mathrm{d}y}\right]\right\}. \tag{78}$$

It was observed in [6] that the presence of the pretwist results in a considerable reduction of the maximum direct normal stress $\sigma_{xx}^D = N_{xx}/h$ (about 23 per cent) for a certain range of the pretwist parameter $\lambda = \beta b^2/h_0$ where h_0 is the (uniform) shell thickness. For a homogeneous isotropic shell of symmetric cross section, we have upon substituting (71) and (77) along with $c_2 = c_4 = 0$ into the third equation of (70)

$$\frac{\sigma_{xx}^D}{\sigma_0} = \frac{xy}{ab}\left\{1+\nu\lambda^2\alpha_1\left(1-\alpha_2\frac{y^2}{b^2}\right)\right\} \tag{79}$$

where $\sigma_0 = Q_s ab/\int_{-b}^{b} y^2 h\,\mathrm{d}y$ is the maximum value of $|\sigma_{xx}^D|$ for a flat sheet, and where

$$\alpha_1 = \frac{h_0^2}{b^4}\frac{\int_{-b}^{b} y^4 C\,\mathrm{d}y}{(1-\nu_M^2)\int_{-b}^{b} D\,\mathrm{d}y}, \qquad \alpha_2 = \frac{b^2\int_{-b}^{b} y^2 C\,\mathrm{d}y}{\int_{-b}^{b} y^4 C\,\mathrm{d}y}. \tag{80}$$

For the cases of rectangular, elliptical, lenticular and diamond shaped cross sections, the following results are obtained,

$$\begin{aligned}
&\frac{h}{h_0} = 1; && \alpha_1 = \tfrac{8}{5}, && \alpha_2 = \tfrac{5}{3},\\
&\frac{h}{h_0} = \sqrt{\left(1-\frac{y^2}{b^2}\right)}; && \alpha_1 = \tfrac{4}{3}, && \alpha_2 = 2,\\
&\frac{h}{h_0} = 1-\frac{y^2}{b^2}; && \alpha_1 = 1, && \alpha_2 = \tfrac{7}{3},\\
&\frac{h}{h_0} = 1-\frac{|y|}{b}; && \alpha_1 = \tfrac{16}{15}, && \alpha_2 = \tfrac{5}{2}.
\end{aligned} \tag{81}$$

On the basis of equation (79), we also observe the following: 1. For $\nu_M\lambda^2 \le 1/\alpha_1(3\alpha_2-1)$, the maximum of σ_{xx}^D is attained at $x = \pm a$ and $y = \pm b$ with

$$\left.\frac{|\sigma_{xx}^D|}{\sigma_0}\right|_{\max} = 1+\nu_M\lambda^2\alpha_1(1-\alpha_2). \tag{82}$$

2. For $\nu_M\lambda^2 \ge 1/\alpha_1(3\alpha_2-1)$, the maximum of σ_{xx}^D is attained at $x = \pm a$ and $y = \pm(1+\nu_M\lambda^2\alpha_1)^{1/2}/(3\nu_M\lambda^2\alpha_1\alpha_2)^{1/2}$ with

$$\left.\frac{|\sigma_{xx}^D|}{\sigma_0}\right|_{\max} = \frac{2(1+\nu_M\lambda^2\alpha_1)^{3/2}}{(27\nu_M\lambda^2\alpha_1\alpha_2)^{1/2}}. \tag{83}$$

For the case $h = h_0$, these results reduce to what has been obtained in [6].

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Абстракт—Обообщаются выведенные раньше результаты для указанных случаев канонической нагрузки скрученных прямоугольных пластинок или пологих еликоидальных оболочек, обладающих прямоугольным поперечным сечением, для определения произвольных заданных изменений поперечной толщины. Настоящий метод вывода оказывается более простым, чем метод выраженный уравнениями Маргерра для поперечных изгибов и функции Эри. Этот метод применяет ирцесс полуобратного решения к системе уравнений равновесия и совместности, для теории пологих оболочек. Особый случай результатов этой работы касается задачи чистого изгиба скрученной пластинки с элиптическим изменением толщины. Результаты для этого случая совпадают с последними результатами, полученными Гудьером и Гриффином, которые пользовались трехмерной теорией упругости.